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Abstract. We examine the dynamics of the Kuramoto model with a new analytical approach. By defining an appropriate set of moments the dynamical equations can be exactly closed. We discuss some applications of the formalism such as the existence of an effective Hamiltonian for the dynamics. We also show how this approach can be used to numerically investigate the dynamical behaviour of the model without finite-size effects.

The study of the dynamical behaviour of systems with a very large number of mutual interacting units is a well debated subject. It is a topic with interest in many different interdisciplinar fields. The cooperation between the members of a population may lead to very rich dynamical situations ranging from chaos, periodicity, phase locking, synchronization to self-organized critical states, just to cite a few [1, 2]. In the presence of disorder such interaction can be frustrated and this yields new types of behaviour. In the realm of disordered systems much work has been devoted to the study of models with relaxational dynamics, for instance spin-glass models [3]. In those cases there exists a Hamiltonian function which governs the dynamics of the system. A large body of information can be obtained by using the tools of statistical mechanics. One of the main results at equilibrium is that the fluctuation–dissipation theorem is obeyed. But it is definitely interesting to study the dynamical behaviour of dissipative systems in the presence of external driving forces.

A simple model of this type was proposed by Kuramoto to analyse synchronization phenomena in populations of weakly nonlinearly coupled oscillators [4]. It has recently become a subject of extensive studies due to its applications to biology, chemistry and physics [5]. The purpose of this paper is to present a new analytical approach to the Kuramoto model based on the definition of a suitable hierarchy of moments. It allows us to reproduce previous known results and, in addition, gives a new insight into the nature of the problem. Here, we will present the method and consider its potential applications leaving detailed analysis for future work. Through this formalism it is possible to analyse some aspects of the model that deserve special attention. As an example, it has been suggested that, under certain conditions, it is possible to define a suitable Hamiltonian function

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from which it is possible to compute stationary properties of the system within the usual thermodynamic formalism such as ground states and universality classes at zero temperature [6] and equilibrium Boltzmann distribution in the more general case at finite temperature [7]. Our method can answer this question in a simple way. We will also show how our approach can be used to numerically investigate the behaviour of the Kuramoto model free of finite-size effects. Particular results will be obtained for the bimodal distribution case.

Our formalism complements other recent theories developed to analyse the Kuramoto model. In particular, it is worthwhile mentioning the order function approach [8] useful for studying properties of the stationary states of the system as well as the critical exponent of the order parameter at the onset of entrainment. Another interesting method was proposed in [9] based on kinetic theory and suitable to deal with questions related to the time dependence of the probability density of the system.

The Kuramoto model is defined by a set of N oscillators whose state can be specified in terms of only one degree of freedom, the phase. Each phase $\{\phi_i; 1 \leqslant i \leqslant N\}$ follows the dynamical equation

$$\frac{\partial \phi_i}{\partial t} = \omega_i - \frac{K_0}{N} \sum_{j=1}^{N} \sin(\phi_i - \phi_j) + \eta_i \tag{1}$$

where ω_i is the intrinsic frequency of the oscillator randomly chosen from a distribution of density $g(\omega)$, K_0 is the strength of the coupling which, as in the original case, we will consider ferromagnetic although more complex situations have been analysed in the literature [10]. Finally, $\eta_i(t)$ denotes a Gaussian-independent white-noise process

$$\langle \eta_i(t)\eta_j(t') \rangle = 2T\delta_{ij}\delta(t - t'). \tag{2}$$

Without any other element there is a competition between the coupling, which tends to synchronize all the oscillators, and the noise (frequencies plus thermal noise) which breaks the coherence. For a critical K_c there is a spontaneous transition from incoherence to a new state where a macroscopic number of units are synchronized.

To solve the dynamics of the Kuramoto model we define the the following set of moments,

$$H_k^m = \frac{1}{N} \sum_{j=1}^{N} \overline{\langle \exp(ik\phi_j) \rangle \omega_j^m} \tag{3}$$

where i is the imaginary unit and k, m are integers in the range $(-\infty, \infty), [0, \infty)$ respectively. The averages $\langle (\cdot) \rangle$ and $\overline{(\cdot)}$ indicate averages over the noise and frequency distributions respectively. The definition of this set of moments is the basis of the new dynamical approach we are proposing. Note that H_k^m is the more natural object we can construct which is invariant under the local transformation $\phi_i \to \phi_i + 2\pi$. This is also the local symmetry of the dynamical equations (1). It is possible to show that the moments H_k^m are self-averaging with respect to the thermal noise†. The equation of motion for H_k^m can easily be derived using equation (3),

$$\frac{\partial H_k^m}{\partial t} = -\frac{K_0 k}{2}(H_{k+1}^m h_{-1} - H_{k-1}^m h_1) - k^2 T H_k^m + ik H_k^{m+1} \tag{4}$$

where we have defined that $h_k = H_k^0$. The term $k^2 T H_k^m$ can be simply obtained using the Gaussian representation for the noise η_i, performing integration by parts and

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† The derivation of this result comes from the fact that a probability density of oscillators can be defined for the Kuramoto model, see Strogatz and Mirollo [5].

using the regularization condition $\frac{\partial \phi(t)}{\partial \eta(t)} = \frac{1}{2}$. In this form the equations are closed because the time operator $\frac{\partial}{\partial t}$ acting on the moment H_k^m always generates new moments of the same type. If we define the time-dependent generating function $g_t(x, y) = \sum_{k=-\infty}^{\infty} \sum_{m=0}^{\infty} \exp(-ikx) \frac{y^m}{m!} H_k^m(t)$ it is easy to check from equation (4) that it satisfies the following differential equation,

$$\frac{\partial g_t}{\partial t} = -\frac{\partial}{\partial x}(A(x, t)g_t) + T \frac{\partial^2 g_t}{\partial x^2} - \frac{\partial^2 g_t}{\partial x \partial y} \tag{5}$$

where

$$A(x, t) = -\frac{K_0}{2i}(\exp(-ix)h_1 - \exp(ix)h_{-1})$$

$$= \frac{K_0}{N} \sum_{j=1}^{N} \sin(x - \phi_j) = K_0 r \sin(\theta - x) \tag{6}$$

and we have expressed $h_1 = h_{-1}^* = r\exp(i\theta)$ where r is the parameter which measures the coherence (synchronization) between oscillators. By substituting equation (3) into the definition of the generating function it is straightforward to check that $g_t(x, y) = \frac{1}{N} \sum_{j=1}^{N} \delta(\phi_j - x) \exp(y\omega_j)$. For $y = 0$ this is nothing else but the probability density of one oscillator having phase x. In this way we recover the results obtained by Bonilla [11] using the path integral formalism. The hierarchy of equations (4) only depends on the time evolution of the moment $h_1(h_{-1} = h_1^*)$ which is the order parameter of the problem. The full set of moments are self-consistently computed using the conditions $h_1(t) = \frac{1}{2\pi} \int_0^{2\pi} \exp(ix)g_t(x, 0) \, dx$ and $H_0^m = \overline{\omega^m}$. In this way, we have reached a dynamical solution of the problem identical to that found in some mean-field glassy models where dynamical equations can be exactly closed [12].

Once we have presented the formulation we present its applications. The method furnishes a clear way to show why a static description based on conventional equilibrium statistical mechanics cannot give reliable information about the long-time properties of the Kuramoto model (such as the existence of stationary states). Let us stress from the beginning that our approach deals with the dynamics of the model once the thermodynamic limit $N \to \infty$ is taken first then the limit $t \to \infty$. Our results are meaningful in this case. In the other dynamical approach one takes the thermodynamic limit $N \to \infty$ after solving the dynamics in the long-time limit $t \to \infty$. This yields quite different results. These considerations are of special interest to the results reported in [6] where the last situation has been analysed. Which of the two approaches is valid relies on the timescales one is able to observe. Because the crossover time which matches the two approaches grows extremely fast with the system size, our approach (which, on the other hand, is the most conventional one) is more suited to investigate what can be observed in macroscopic systems in realistic timescales.

Let us consider the following Hamiltonian function,

$$\mathcal{H}_{\text{eff}} = -\frac{K_0}{N} \sum_{i<j} \cos(\phi_i - \phi_j) - \sum_i \omega_i \phi_i \tag{7}$$

where the phases ϕ_i are restricted to the interval $[-\pi, \pi)$ in order $\mathcal{H}_{\text{eff}}$ to be bounded from below. This Hamiltonian function is the only candidate which generates equations (1) in the Langevin dynamics $\dot{\phi}_i = -\frac{\partial \mathcal{H}_{\text{eff}}}{\partial \phi_i} + \eta_i$. We will show that the equilibrium solutions of (7) are not stationary solutions of the dynamics equation (1) even at zero temperature. To prove this result we compute the partition function of equation (7) at temperature $T = \frac{1}{\beta}$ and evaluate the moments $H_k^m(eq)$ in equilibrium. The computations are quite simple since

the disorder in $\mathcal{H}_{\text{eff}}$ is only site dependent. For sake of simplicity we will consider here the case in which the disorder distribution is symmetric ($g(\omega) = g(-\omega)$). It is easy to obtain the equilibrium values of the different moments. We obtain $H_k^m(eq) = F_k^m(eq)\exp(ik\theta)$ where θ is an arbitrary phase† and the $F_k^m(eq)$ are given by

$$F_k^m(eq) = \overline{\omega^m R_k^\omega(\beta K_0 r)} \qquad (8)$$

where the $\overline{A(\omega)} = \int d\omega\, g(\omega) A(\omega)$ is the average over the frequency distribution and $R_k^\omega(x) = \frac{J_k^\omega(x)}{J_0^\omega(x)}$. The $J_k^\omega(x)$ are generalized Bessel-like functions defined by,

$$J_k^\omega(x) = \int_0^{2\pi} d\phi\, \exp(ik\phi + x\cos(\phi) + \beta\omega\phi) \qquad (9)$$

where r is fixed by the condition $r = F_1^0(eq) = \overline{R_1^\omega(\beta K_0 r)}$.

To check if the $F_k^m(eq)$ are stationary solutions of the dynamics we plug them into equation (4) and use the recursion relation,

$$J_{k+1}^\omega(x) = J_{k-1}^\omega(x) - \frac{2(k - i\beta\omega)}{x} J_k^\omega(x) - \frac{2i\exp(x)}{x}(\exp(2\pi\beta\omega) - 1) \quad (10)$$

obtaining after some manipulations,

$$\frac{\partial F_k^m(eq)}{\partial t} = ikv_m \qquad (11)$$

where

$$v_m = T\exp(\beta K_0 r)\frac{\overline{\omega^m(\exp(2\pi\beta\omega) - 1)}}{J_0^\omega(\beta K_0 r)}. \qquad (12)$$

Due to the symmetry property of the $g(\omega)$ it is easy to check that v_m always vanishes for m even. But it never does for odd m. In the high-temperature regime $\beta \to 0$ it is possible to show that $v_{2m-1} = \overline{\omega^{2m}} + \mathrm{O}(\beta^3)$. In the limit $T \to 0$ it is easy to check that $v_{2m-1} = \overline{\omega^{2m}} + \mathrm{O}(\frac{1}{\beta})$. The general result is that odd m-moments violate stationarity‡. We conclude that the time derivative of the equilibrium moments $H_k^m(eq)$ with odd m get an imaginary contribution proportional to $i = \exp(i\frac{\pi}{2})$ which is transverse in the complex plane to the $H_k^m(eq)$ itself. Note that the term v_m is the angular velocity or time derivative of the global phase for all the moments which only depends on the number m. In the case of the bimodal distribution $g(\omega) = \frac{1}{2}(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$ all moments reduce to two different moments (see below) depending if m is even or odd. In figure 1 we show v_{odd} as a function of the temperature in the bimodal distribution for different values of K_0§. This proves that equilibrium states of $\mathcal{H}_{\text{eff}}$ at finite temperature and also the ground states of $\mathcal{H}_{\text{eff}}$ are not stationary states of the dynamics. Note that we cannot discard the fact that local minima (but not global) of $\mathcal{H}_{\text{eff}}$ at zero temperature are fixed points of the dynamics.

Now we want to show how equations (4) can be used as a powerful tool to investigate the dynamical behaviour of the Kuramoto model. We will consider the case of the bimodal distribution because the phase diagram of the model is very rich. We have checked the situation for other distributions of frequencies and the results have always been very

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† $\mathcal{H}_{\text{eff}}$ only changes by a constant $\Phi\sum_i \omega_i$ if all the phases ϕ_i are changed to $\phi_i + \Phi$.
‡ An exception to this rule are the moments H_0^m, among them the average frequency, because $k = 0$ and therefore the right-hand side of equation (11) vanishes.
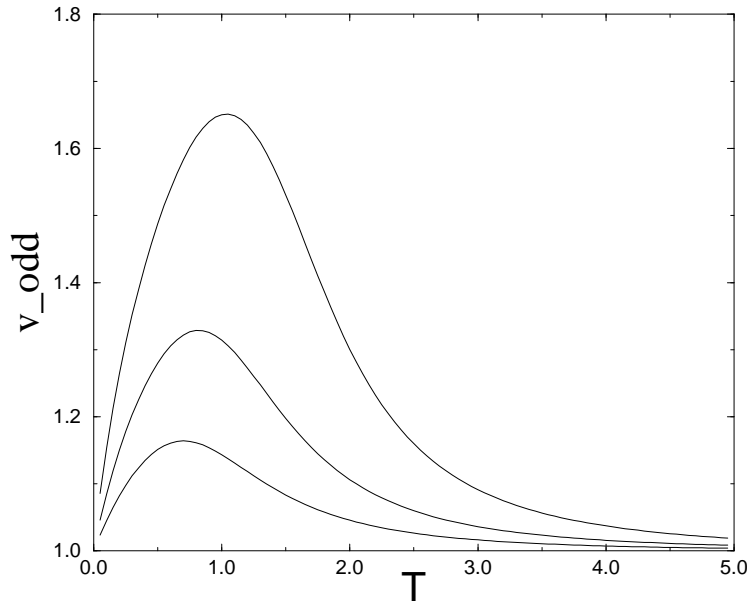§ In the rest of this paper and without loss of generality we will take $\omega_0 = 1$.

Figure 1. v_{odd} as a function of T for the bimodal distribution for three values of $K_0 = 0.5, 1, 2$ from bottom to top. The cusp in v_{odd} moves to higher temperatures as K_0 increases.

satisfactory. In this case the full set of moments H_k^m reduces to two different sets, $f_k = H_k^m$ for m even and $g_k = H_k^m$ for m odd. The full set of dynamical equations read in this case,

$$\frac{\partial f_k}{\partial t} = -\frac{K_0 k}{2}(f_{k+1}f_{-1} - f_{k-1}f_1) - k^2 T f_k + \mathrm{i}k g_k \tag{13}$$

$$\frac{\partial g_k}{\partial t} = -\frac{K_0 k}{2}(g_{k+1}f_{-1} - g_{k-1}f_1) - k^2 T g_k + \mathrm{i}k f_k. \tag{14}$$

By defining

$$\rho_+(x) = \tfrac{1}{2}\sum_{k=-\infty}^{\infty} , \exp(-\mathrm{i}kx)(f_k + g_k)$$

and

$$\rho_-(x) = \tfrac{1}{2}\sum_{k=-\infty}^{\infty} \exp(-\mathrm{i}kx)(f_k - g_k)$$

we observe that these are the probability densities of having one oscillator with phase x and natural frequencies $+1$ and -1 respectively. By adding and substracting equations (13) and (14) we obtain dynamical equations for two sets of dynamical hierarchies, each set characterized by a population of oscillators with a given natural frequency ($+1$ or -1). Note that the two sets of oscillators are also coupled to each other through the terms $\mathrm{i}k g_k$ and $\mathrm{i}k f_k$ in (13) and (14). These equations can also be used to analytically compute stationary states and perform stability analysis as has been done in [13]. In this last case it can be shown that the fundamental model $k = 1$ decouples form the rest of the modes in a natural way. Here we follow a different strategy and use the method to numerically solve the set of equations for a given number $2L+1$ of terms in the hierarchy $\{f_k, g_k; -L \leqslant k \leqslant L\}$. We stress that L is not the number of oscillators in the system which is already infinite from the beginning.
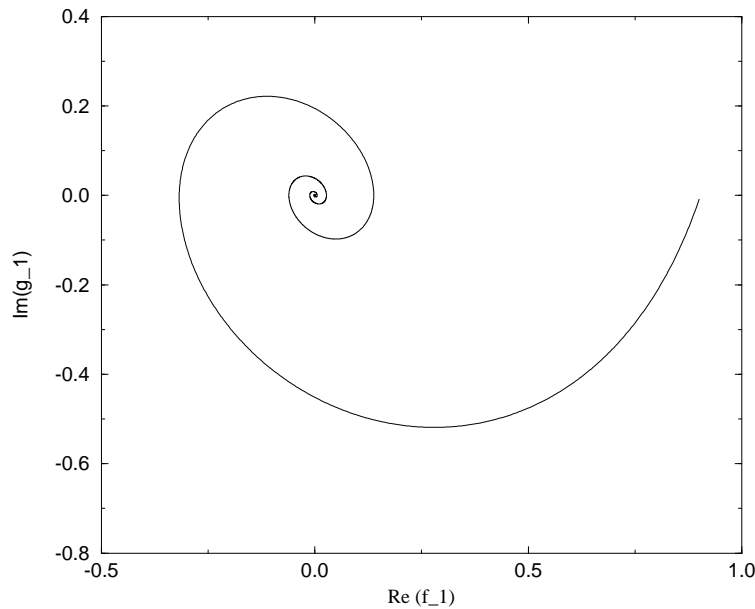
Figure 2. Trajectory of the system in the $(\text{Re}(f_1), \text{Im}(g_1))$ plane at $T = 0.5$, $K_0 = 1$ in the incoherent regime.

To reduce possible dependences on the value of L we consider periodic boundary conditions $f_{L+1} = f^*_{-L}$, $f_{-L-1} = f^*_L$'s and do the same for the g's. In our numerical solution we have taken $L = 100$ and have checked that the results are the same by including more terms in the hierarchy. Equations have been solved using a second-order Euler algorithm. We have also started from an initial condition of the type shown in equation (8) with $\theta = 0$ in order to check they are not stationary solutions. Depending on the values of the parameters K_0 and T there are different regimes [13].

In figures 2–4 we show the trajectory of the system in the plane $(\text{Re}(f_1), \text{Im}(g_1))$ for three different regimes (the incoherent, critical and coherent regimes). Note that according to equation (12) all the trajectories depart from $\text{Im}(g_1)$ in the direction $-i = \exp(\frac{3\pi}{2})$. The first regime (figure 2) corresponds to the region where the incoherent solution is stable. In this case the order parameter r $(r = (f_1 f^*_1)^{\frac{1}{2}})$ oscillates with an amplitude which decays to zero exponentially in time. The second regime is shown in figure 3 and corresponds to the critical boundary line $T = \frac{K_0}{4}$ where the incoherent solution becomes unstable. In this case r oscillates and its amplitude decays to zero algebraically like $t^{-\frac{1}{2}}$ as expected for mean-field models at the critical point. In the region $T < \frac{K_0}{4}$ the incoherent solution is unstable and the system reaches a oscillating stationary solution (see figure 4) in agreement with the results analytically found by Bonilla *et al* [13, 14]. Note that these types of solutions cannot be computed from any theory based on an effective Hamiltonian (EH) like that given by equation (7). Therefore, the assumption of the existence of an effective Hamiltonian not only implies considering solutions which are not stationary but also to miss another set of them that are explicitly time dependent. We have also observed the existence of stationary fixed points for enough large values of K_0 as expected when the ferromagnetic coupling is strong enough. In this case the equilibrium solution within the EH approach albeit incorrect is closer to the true stationary one (in the limit $K_0 \to \infty$ the EH approach is recovered).
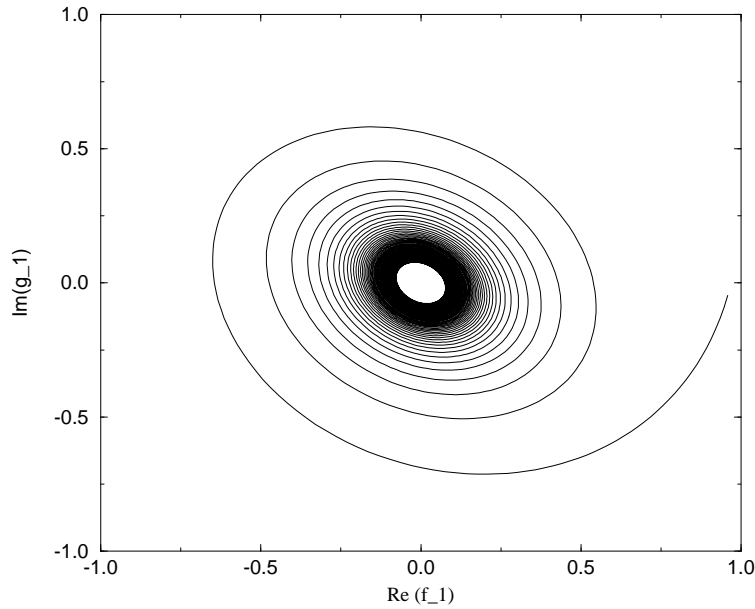
Figure 3. The same as in figure 2 at $T = 0.25$, $K_0 = 1$ in the critical boundary. The area of the central hole decreases like $1/t$.
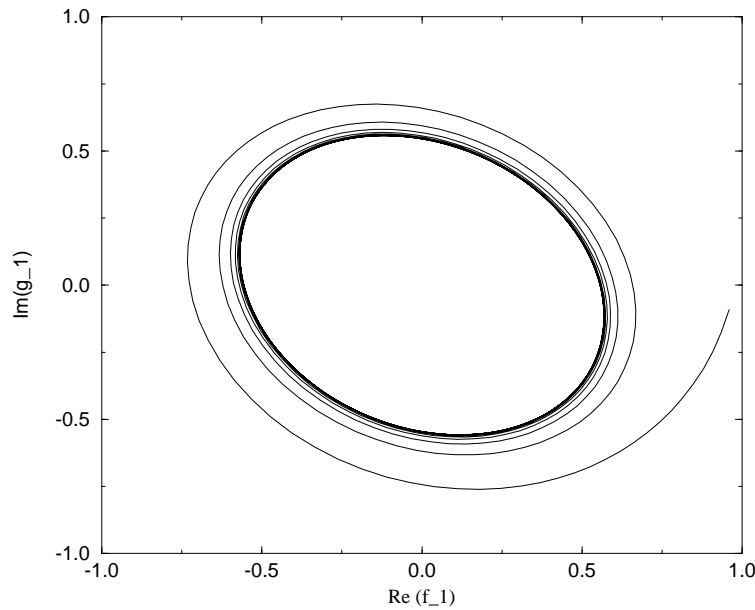

Figure 4. The same as in figure 2 at $T = 0.05$, $K_0 = 1$ in the synchronized regime.

Finally, a comparison between the present theory (equations (13) and (14)) for the bimodal case) and the Brownian simulations is shown in figure 5. Simulations have been performed by solving equation (1) with a Euler method with a time step $\delta t = 0.005$ and for a population of $N = 50\,000$ oscillators. Despite some small differences there is remarkable agreement
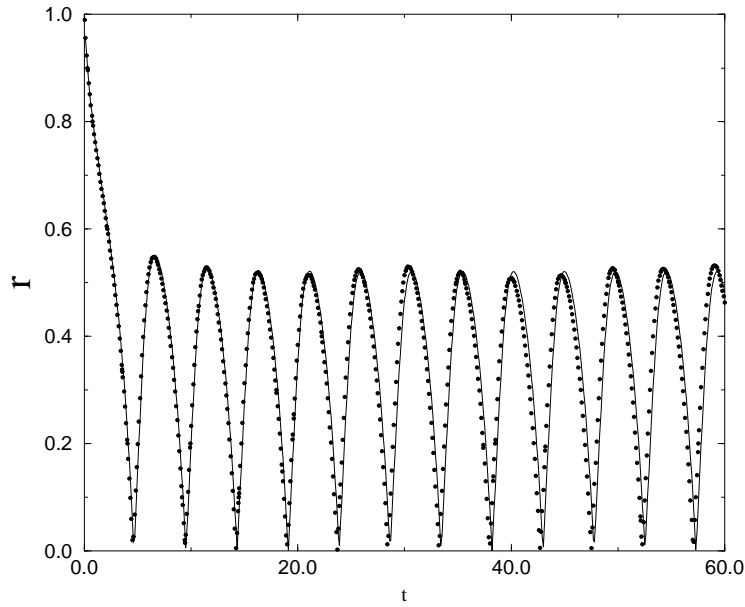
Figure 5. Analytical solution (curves) versus Brownian simulations (points) in the oscillationg regime with parameters $T = 2.5$, $K_0 = \frac{1}{4}$. Brownian simulations were performed with 50 000 oscillators and one realization of the noise.

between the simulation and analytical results.

In summary, we have presented an approach to the analytical solution of the Kuramoto model which is simple in its formulation and suitable for analytic and numerical computations. We have shown that the EH approach fails to predict the stationary states of the system as well as the ground states of the energy function equation (7). We can give an explanation of this result. It has been suggested [6] that if a minimum of the energy function equation (7) can be localized in the interior of the region $[-\pi, \pi)$ then this should be asymptotically stable. In the Kuramoto model it seems that this condition is indeed not satisfied, at least for the ground states of equation (7). As shown in equation (12) the ground states of equation (7) are not stationary states of the dynamics. The quantitative violation of the stationarity property has also been analytically computed in equation (12). The reason for the discrepancy of our results with those reported in [6] relies on how the order of the limits $N \to \infty$ and $t \to \infty$ are taken. The limit $t \to \infty$ particularly concerns the properties of the stationary states. When the limit $N \to \infty$ is taken first then all ground states of $\mathcal{H}_{\text{eff}}$ become dynamically unstable. The reason is that in this limit a finite density of oscillators in the ground state touch the boundaries where the effective Hamiltonian equation (7) is discontinuous. Obviously this is not true if the limit $t \to \infty$ is taken first. Then, for finite N, the phases θ_i of the oscillators do not touch the borders (because it has zero measure) and the ground states are stationary. As said before, the order of limits (first $N \to \infty$ and later $t \to \infty$) is the one expected to describe the relevant asymptotic dynamics. We have used equations (4) to investigate the dynamical behaviour of the Kuramoto model free of finite-size effects. Particular results have been obtained for the bimodal distribution but the method can be generally applied to any other distribution. It would be very interesting to use this approach to study the spectrum of correlation and response functions of the model as well as to investigate the presence of other dynamical regimes.

Acknowledgments

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